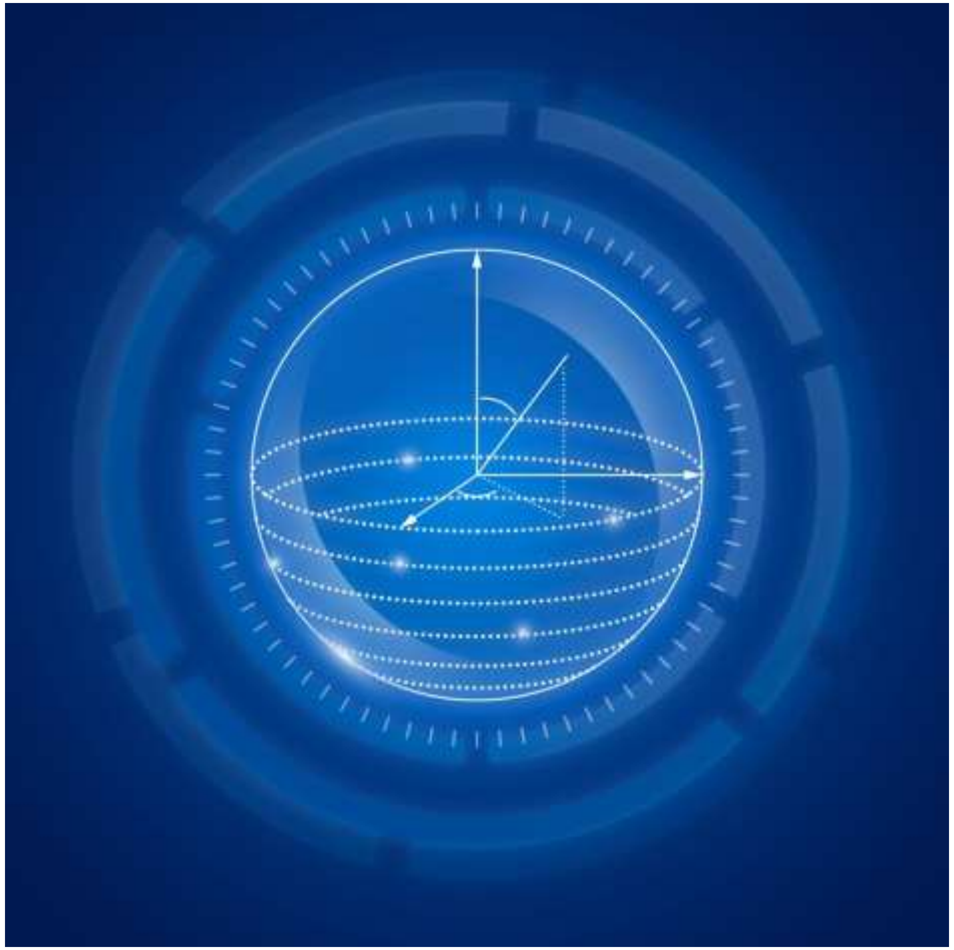


Vectors and coordinates system in space

Vectors in space
Introduction

The notion of vectors is introduced mainly in grade 10 class.
Same in space.

- ✓ Every two points defined a unique vector
- ✓ Notation: $\overrightarrow{AB}$ or $\vec{u}$
- ✓ Direction (Line of action): (AB)
- ✓ Sense: From A to B
- ✓ Magnitude: $\left\|\overrightarrow{AB}\right\| = AB$

Vectors in space

Introduction

Remarks

- ✓ $\overrightarrow{AA} = \vec{0}$
- ✓ $\overrightarrow{AB} = -\overrightarrow{BA}$
- ✓ $\vec{0}$ is a vector having the same direction as every vector

Vectors in space

Introduction

Equal vectors

✓ $\vec{u} = \vec{v}$

- Same direction
- Same sense
- Same norm

If $\overrightarrow{AB} = \overrightarrow{CD}$, then ABDC is a parallelogram

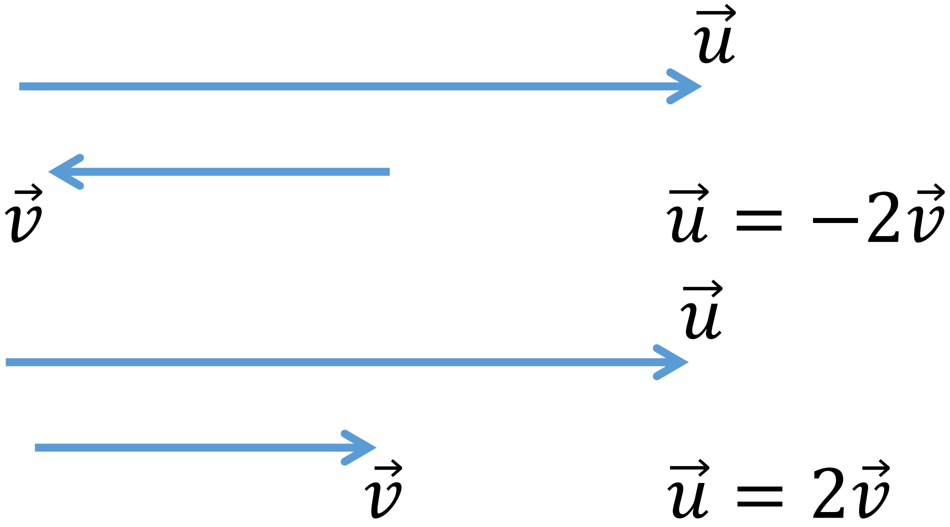
Vectors in space
Introduction

Opposite vectors

- ✓ $\vec{u} = -\vec{v}$
  - Same direction
  - Opposite sense
  - Same norm

Collinear vectors

- ✓ $\vec{u} = k\vec{v}\ \ k \in \mathbb{R}$
  - Same direction
  - If k > 0, same sense
    If k < 0, opposite sense
  - $||\vec{u}|| = |k| \times ||\vec{v}||$

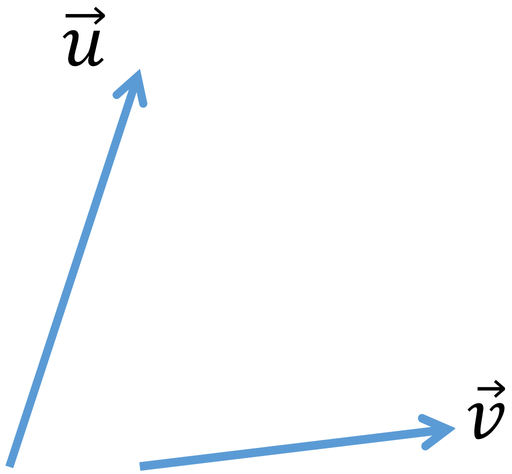


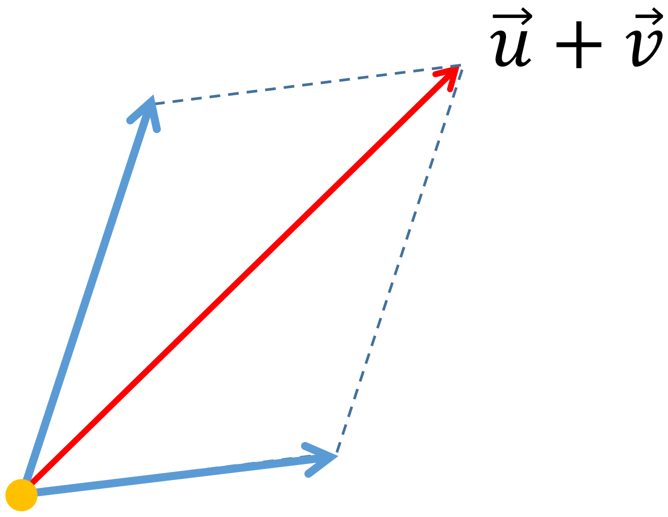
Vectors in space

Introduction

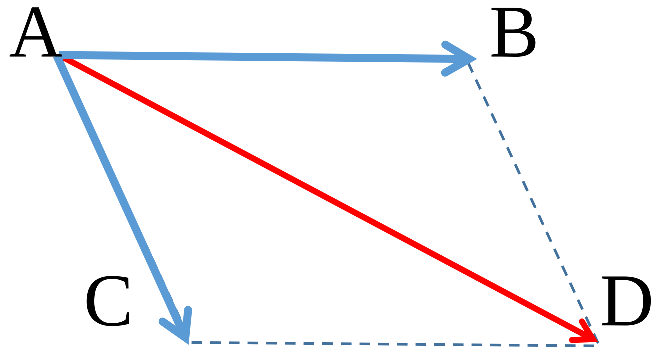
Sum of two vectors

✓ $\vec{u} + \vec{v}$ is a vector





✓ If ABDC is a parallelogram, then $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$ and vice versa

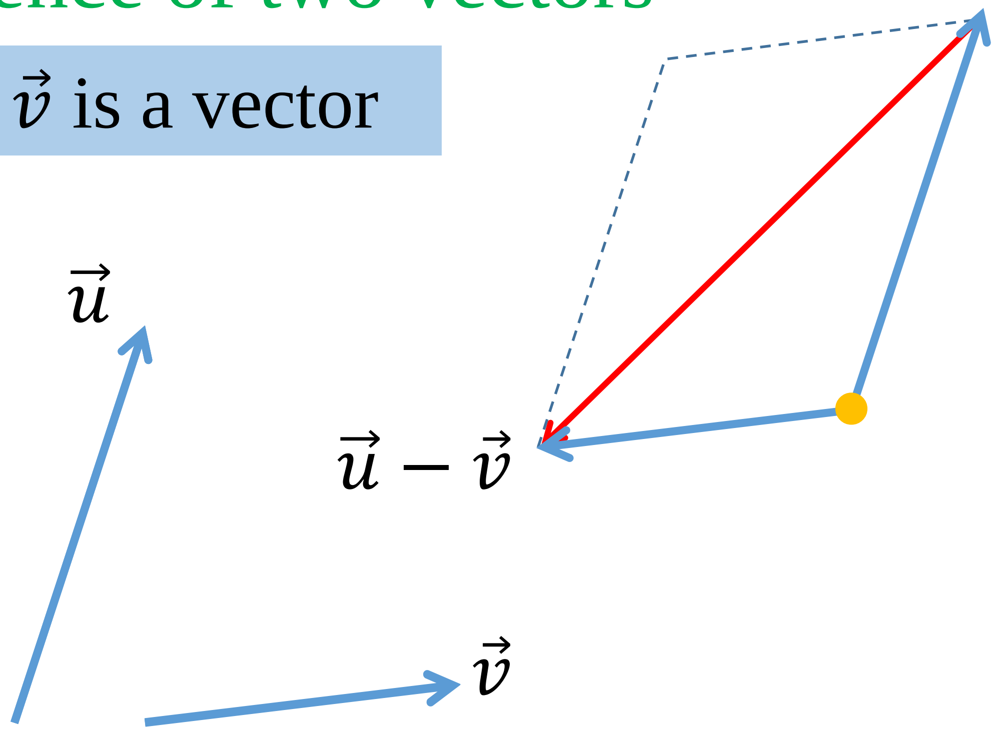


Vectors in space

Introduction

Difference of two vectors

✓ $\vec{u} - \vec{v}$ is a vector



Vectors in space

Introduction

Remarks

a, b are two real numbers, $\vec{u}$ and $\vec{v}$ are two any vectors.

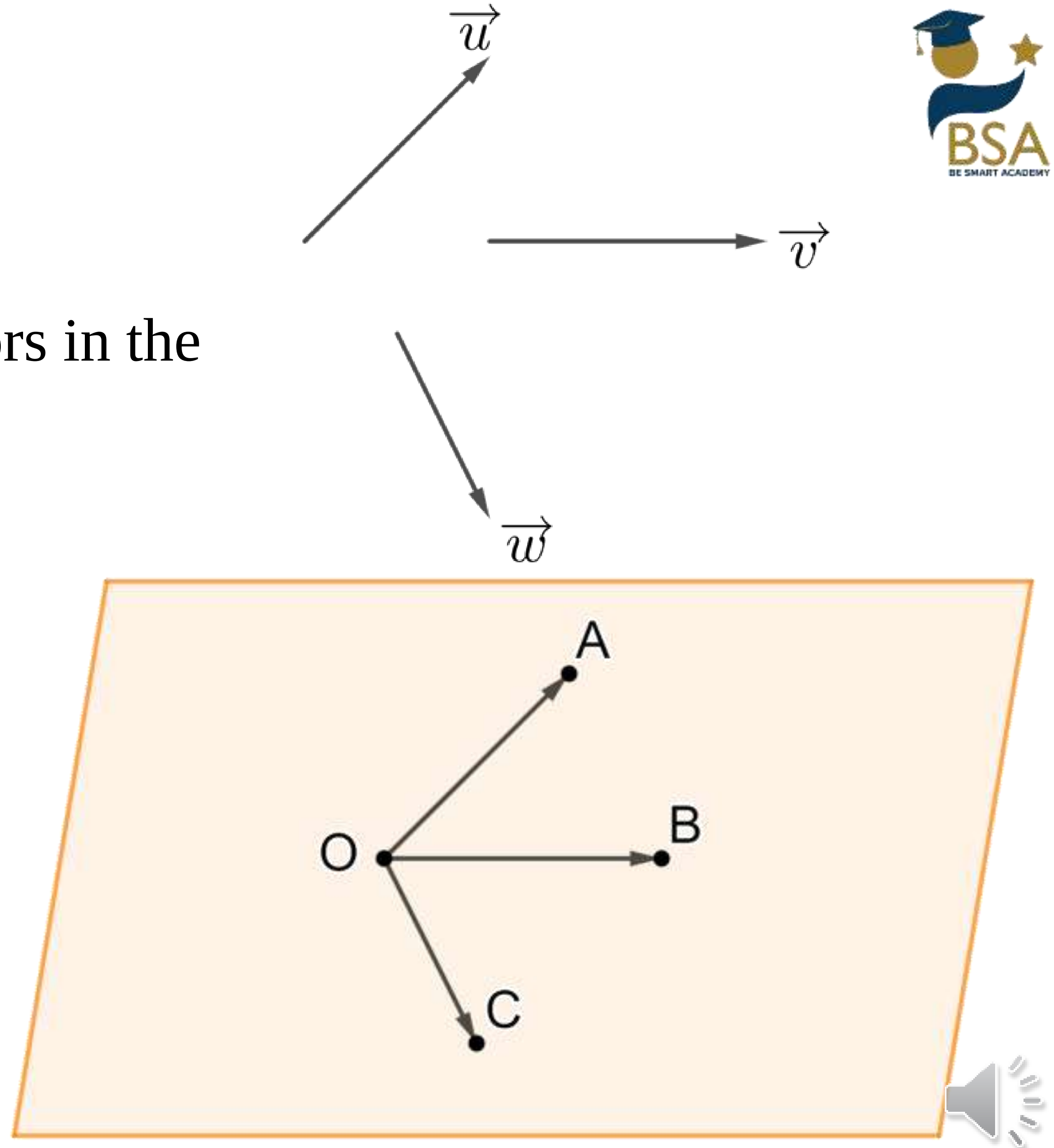
- ✓ $(a+b)\vec{u} = a\vec{u} + b\vec{u}$
- ✓ $a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
- ✓ $a(b\vec{u}) = (ab)\vec{u}$

Vectors in space
Coplanar vectors

Definition

$\vec{u}, \vec{v}$ and $\vec{w}$ are three non collinear vectors in the space.

$\vec{u}$, $\vec{v}$ and $\vec{w}$ are said coplanar when an arbitrary point O and the three points defined by: $\overrightarrow{OA} = \vec{u}$, $\overrightarrow{OB} = \vec{v}$ and $\overrightarrow{OC} = \vec{w}$ are coplanar:
O; A; B and C belong to the same plane.



Vectors in space

Coplanar vectors

Remarks

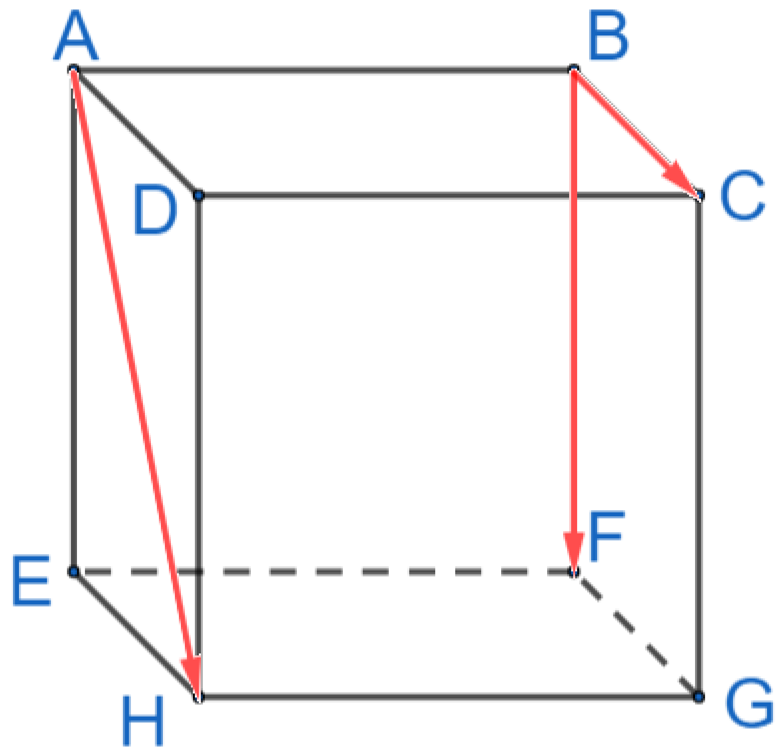
❶ If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar, then exist 2 non zero real numbers a and b such that $\vec{w} = a\vec{u} + b\vec{v}$ (and vice versa)

❷ To prove that 4 points A, B, C and D are coplanar, it is sufficient to prove that 3 vectors of these 4 points are coplanar.

❸ If two of the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ are collinear, then the 3 vectors are coplanar.

Vectors in space
Coplanar vectors

Example



$\overrightarrow{AH} = \overrightarrow{BG}$

B, C, G and F are coplanar

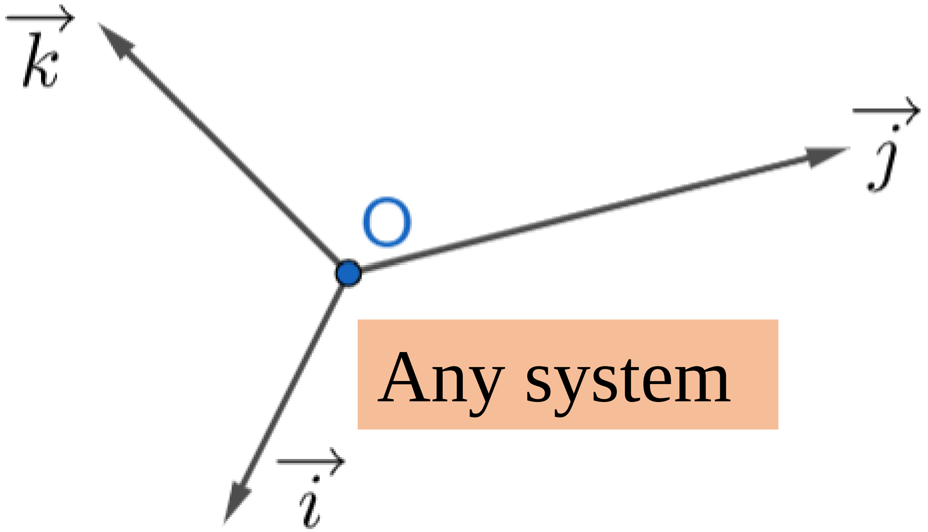
So $\overrightarrow{BC}$, $\overrightarrow{BF}$ and $\overrightarrow{AH}$ are coplanar.

System of coordinates

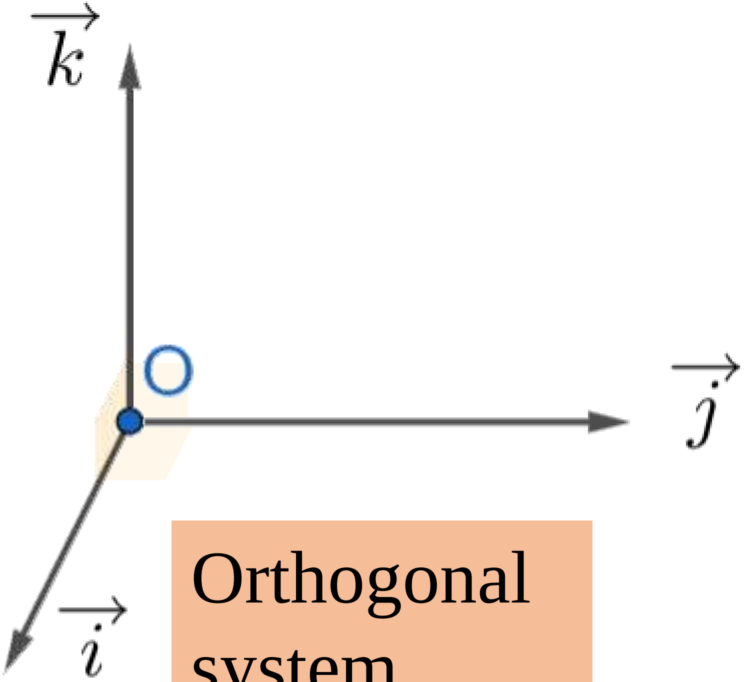
Basis

Definition

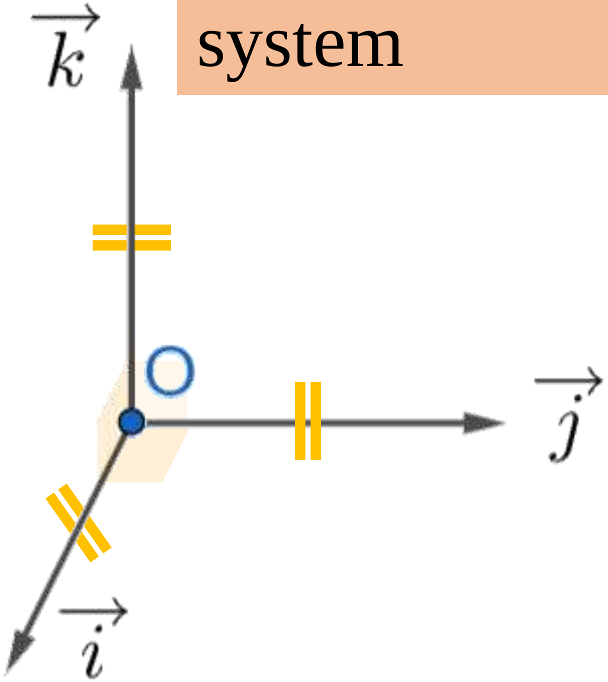
Any non coplanar triple of vectors form a basis in space.



Any system



Orthogonal system

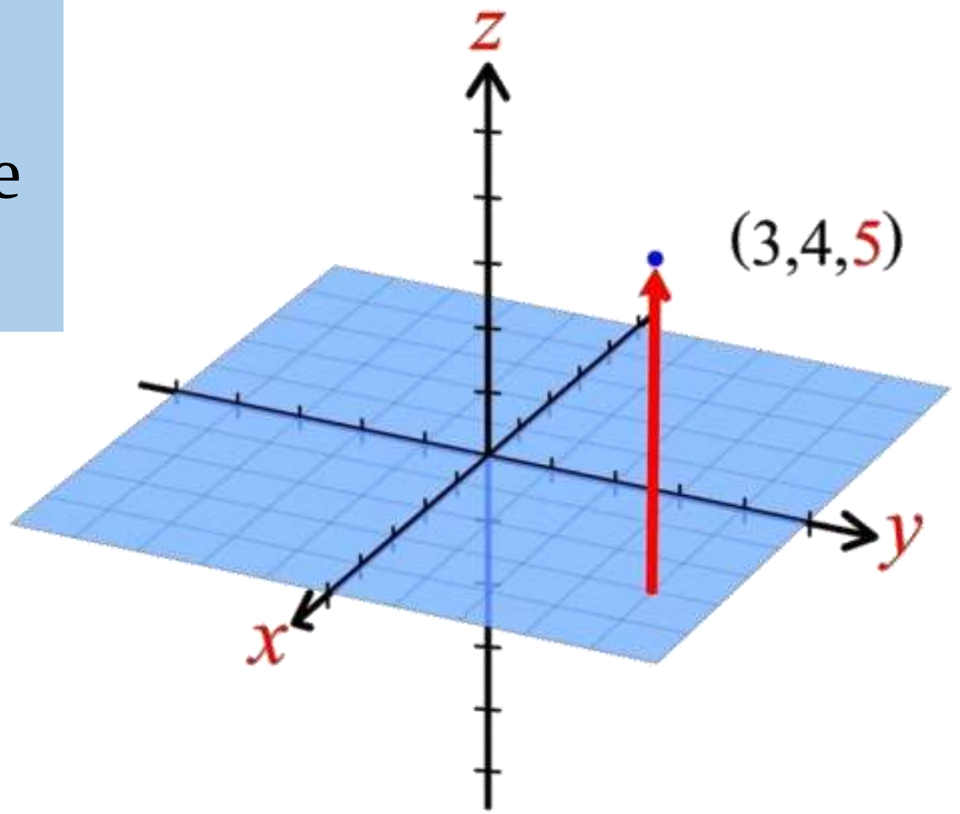


Orthonormal system

System of coordinates
Coordinates of a point

Definition

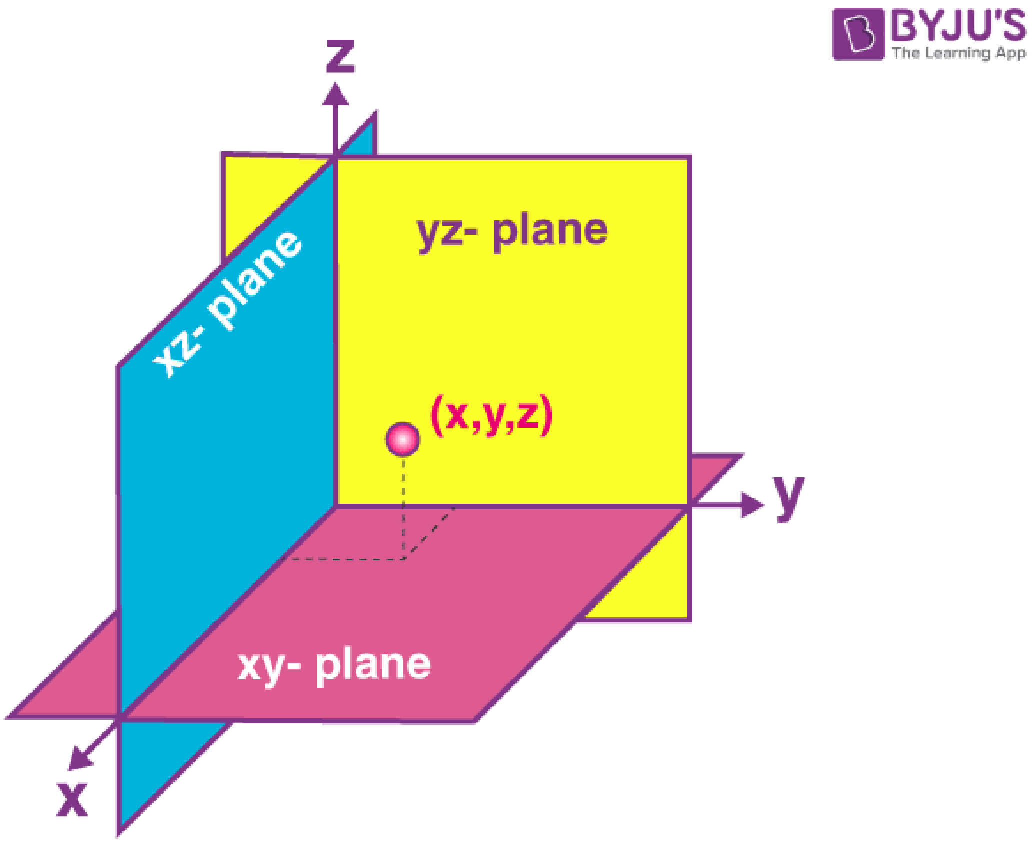
Consider the orthonormal system $(O;\vec{\imath}\,;\vec{\jmath}\,;\vec{k})$.
M is any point of the space.
For every point M in the space, we can associate a triplet $(x;y;z)$ such that: $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$



x is abscissa of M
y is ordinate of M
z is elevation of M

System of coordinates
Coordinates of a point

Remarks:



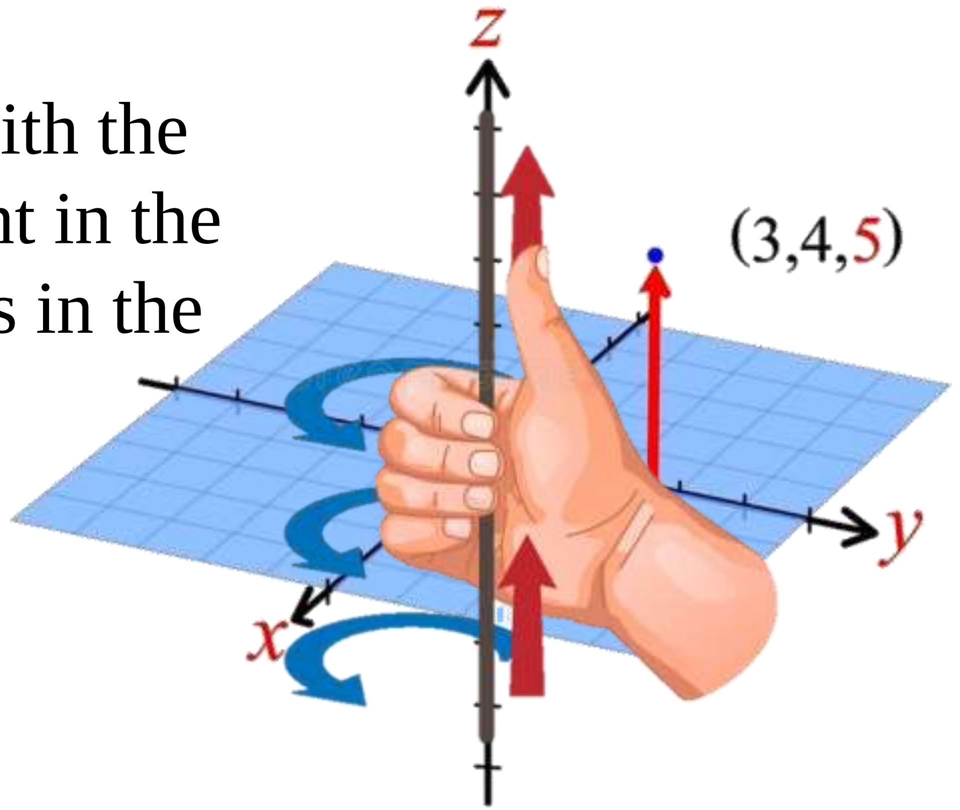
- ✓ M is a point in xy-plane: $z_M = 0$
- ✓ M is a point of xz-plane: $y_M = 0$
- ✓ M is a point on yz-plane: $x_M = 0$
- ✓ M is a point on xAxis: $y_M = z_M = 0$
- ✓ M is a point on yAxis: $x_M = z_M = 0$
- ✓ M is a point on zAxis: $x_M = y_M = 0$

System of coordinates
Coordinates of a point

Remarks:

The sense of the elevation axis (z’z) is determine by right hand rule.

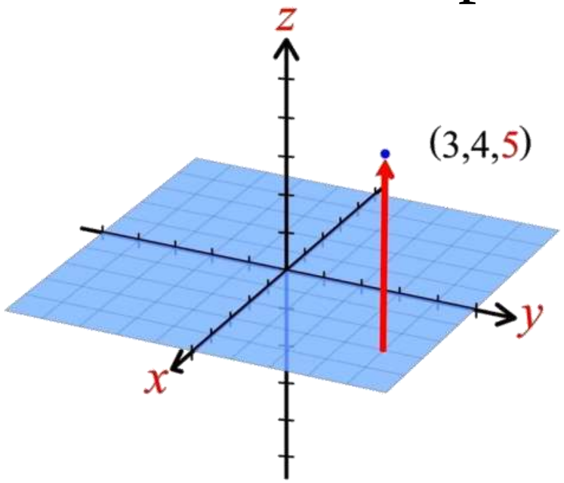
If we take our right hand and align the fingers with the positive x-axis, then curl the fingers so they point in the direction of the positive y-axis, our thumb points in the direction of the positive z-axis.

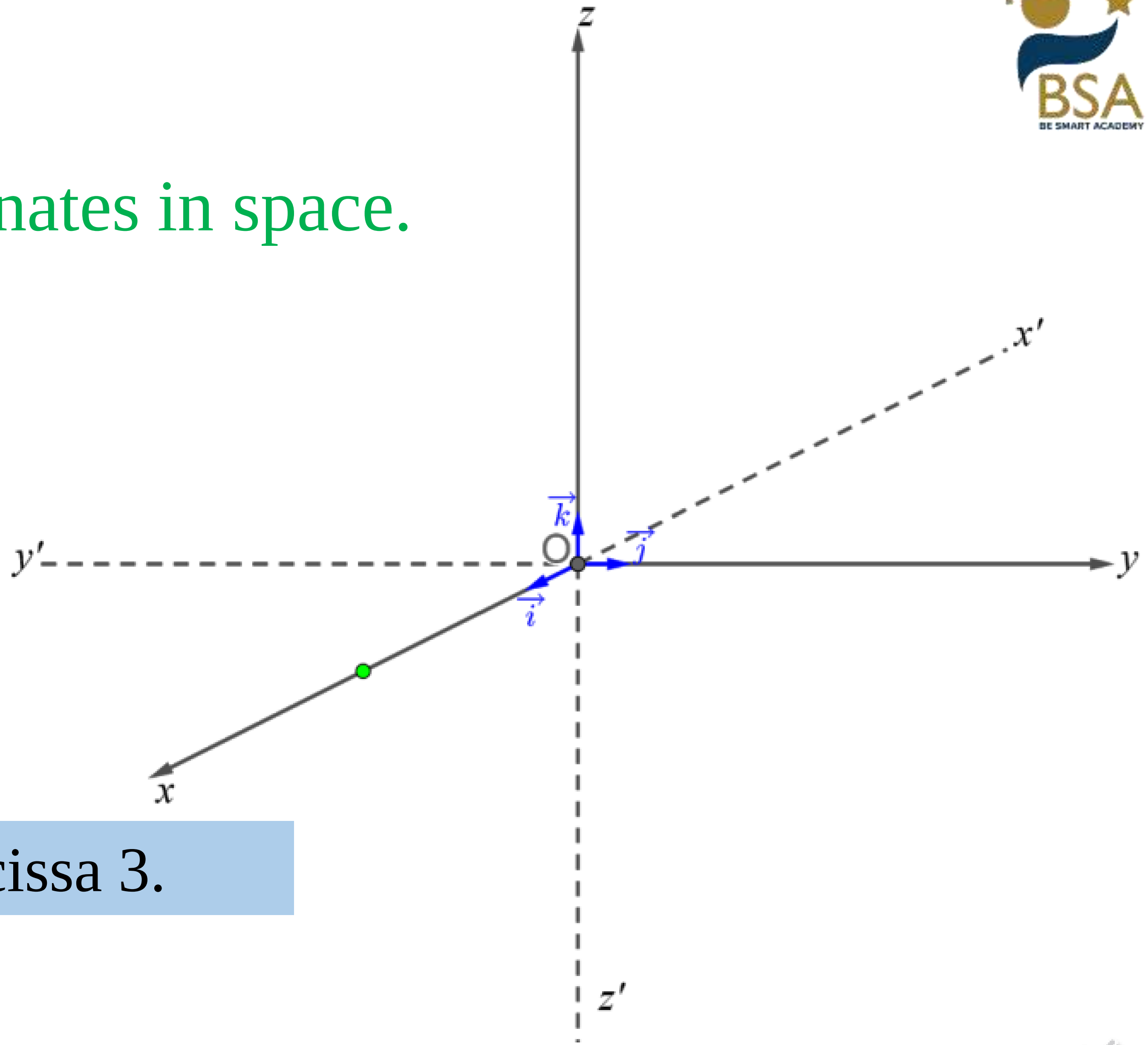


System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



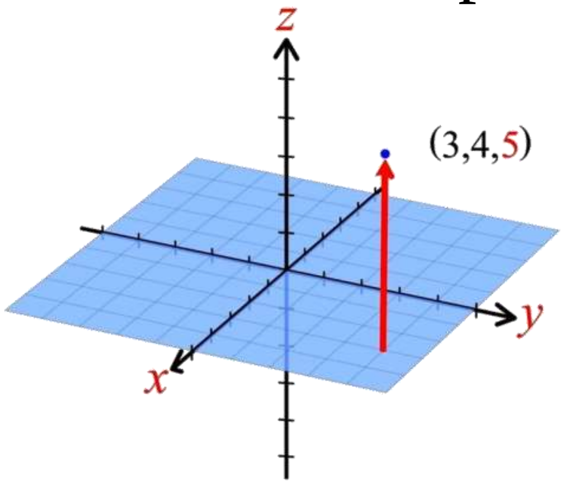


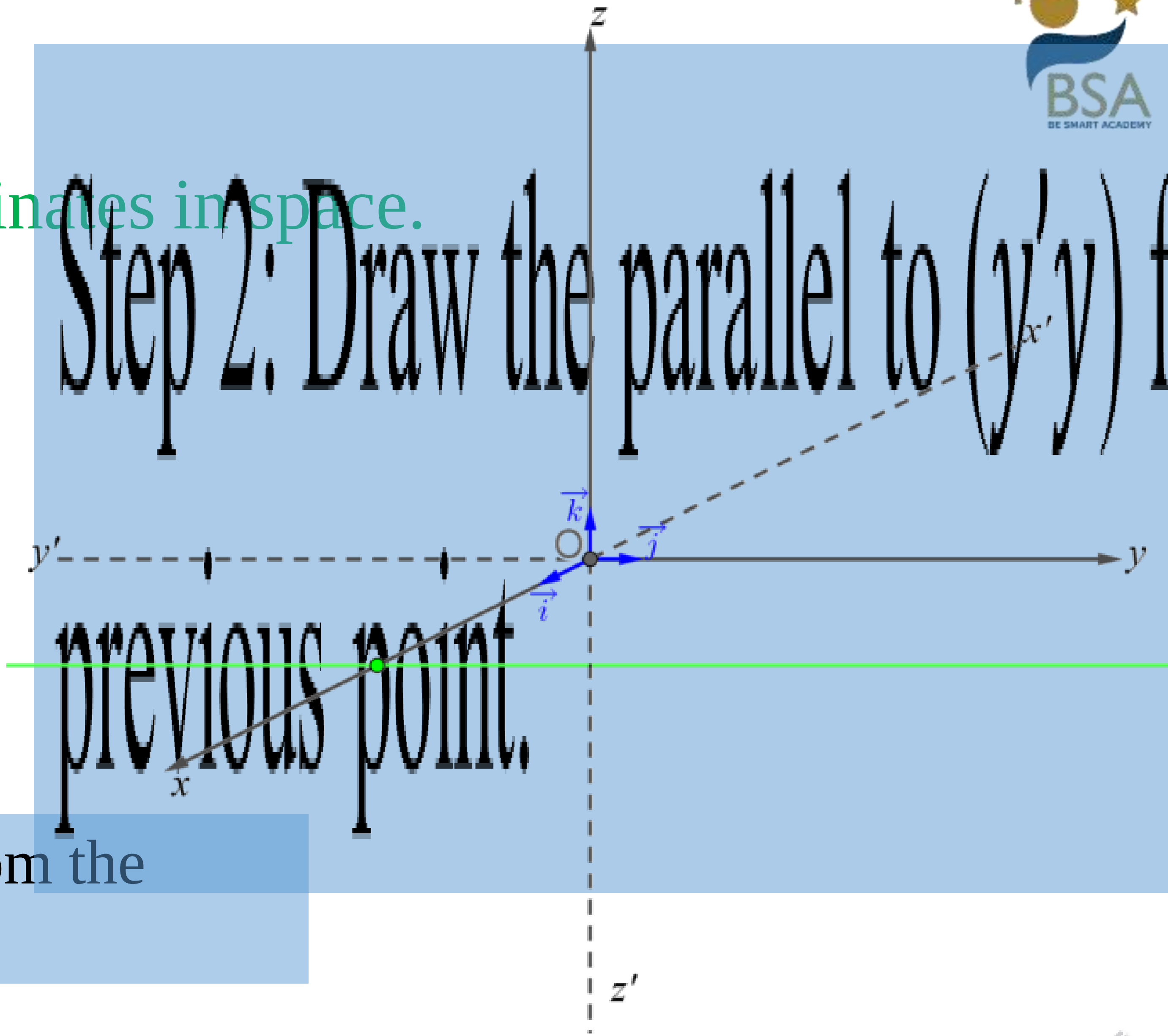
Step 1: Plot on $(x'x)$ the point of abscissa 3.

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)





Step 2: Draw the parallel to (y’y) from the previous point.

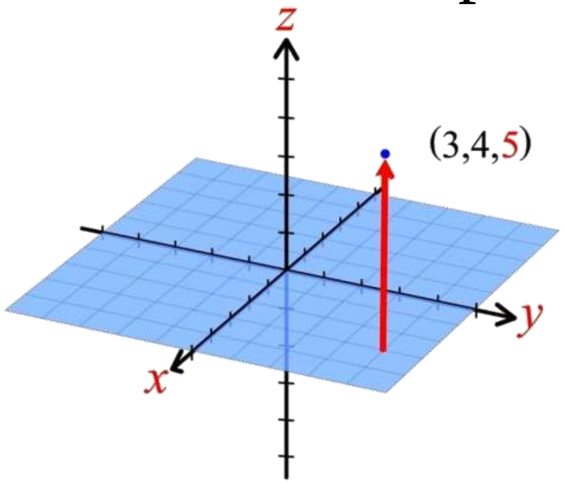
Step 2: Draw the parallel to (y'y) f
previous point.

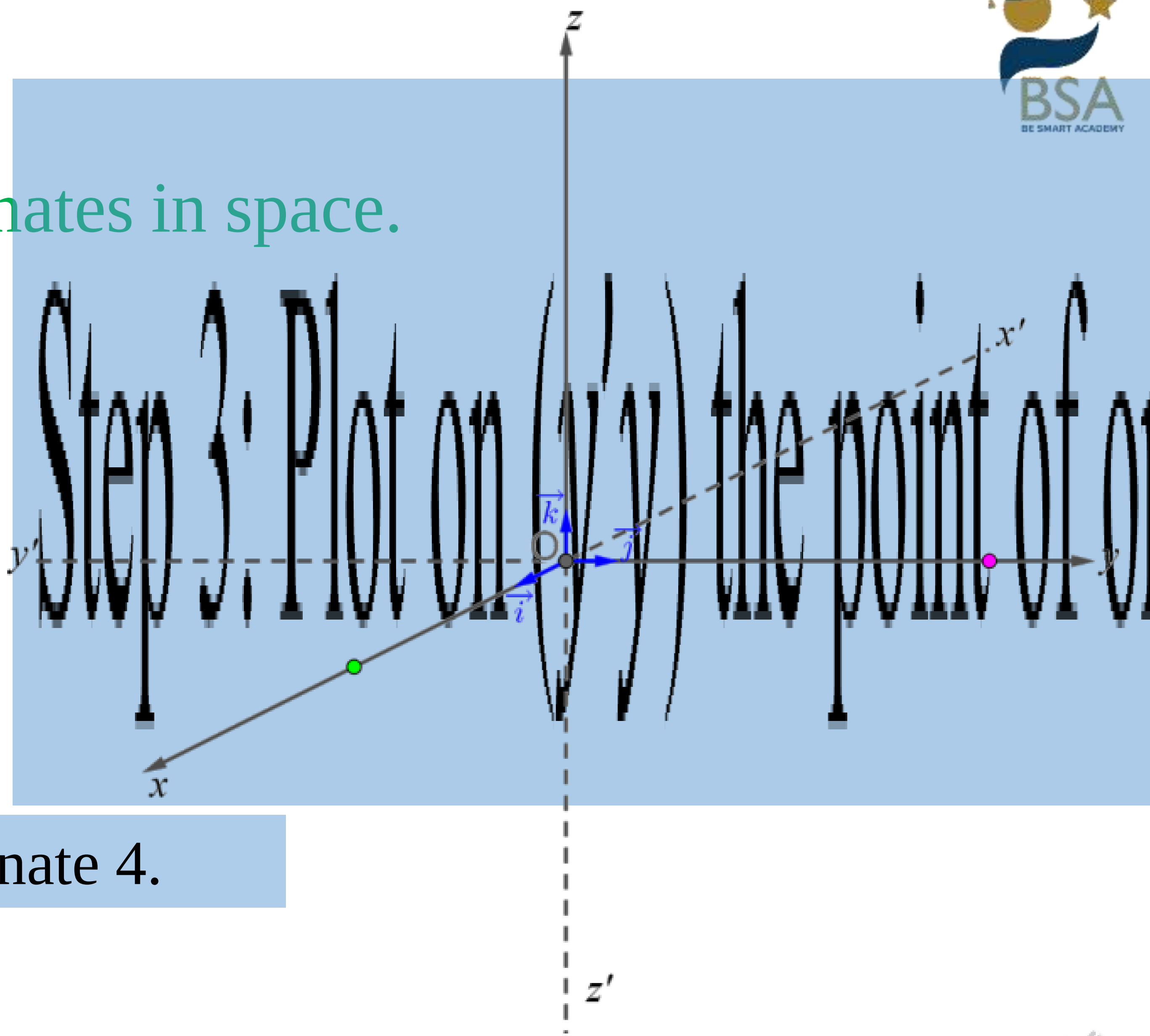
System of coordinates

Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



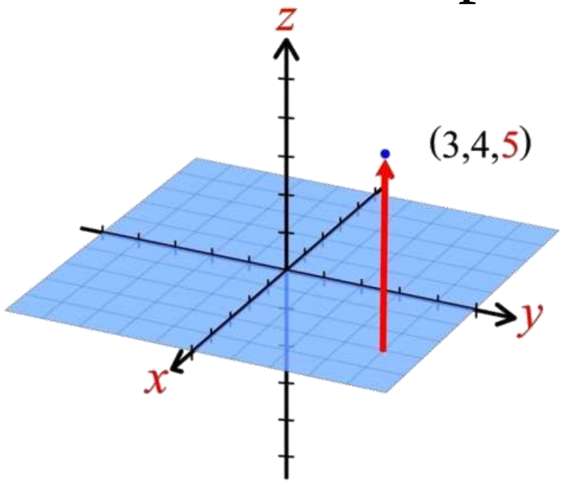


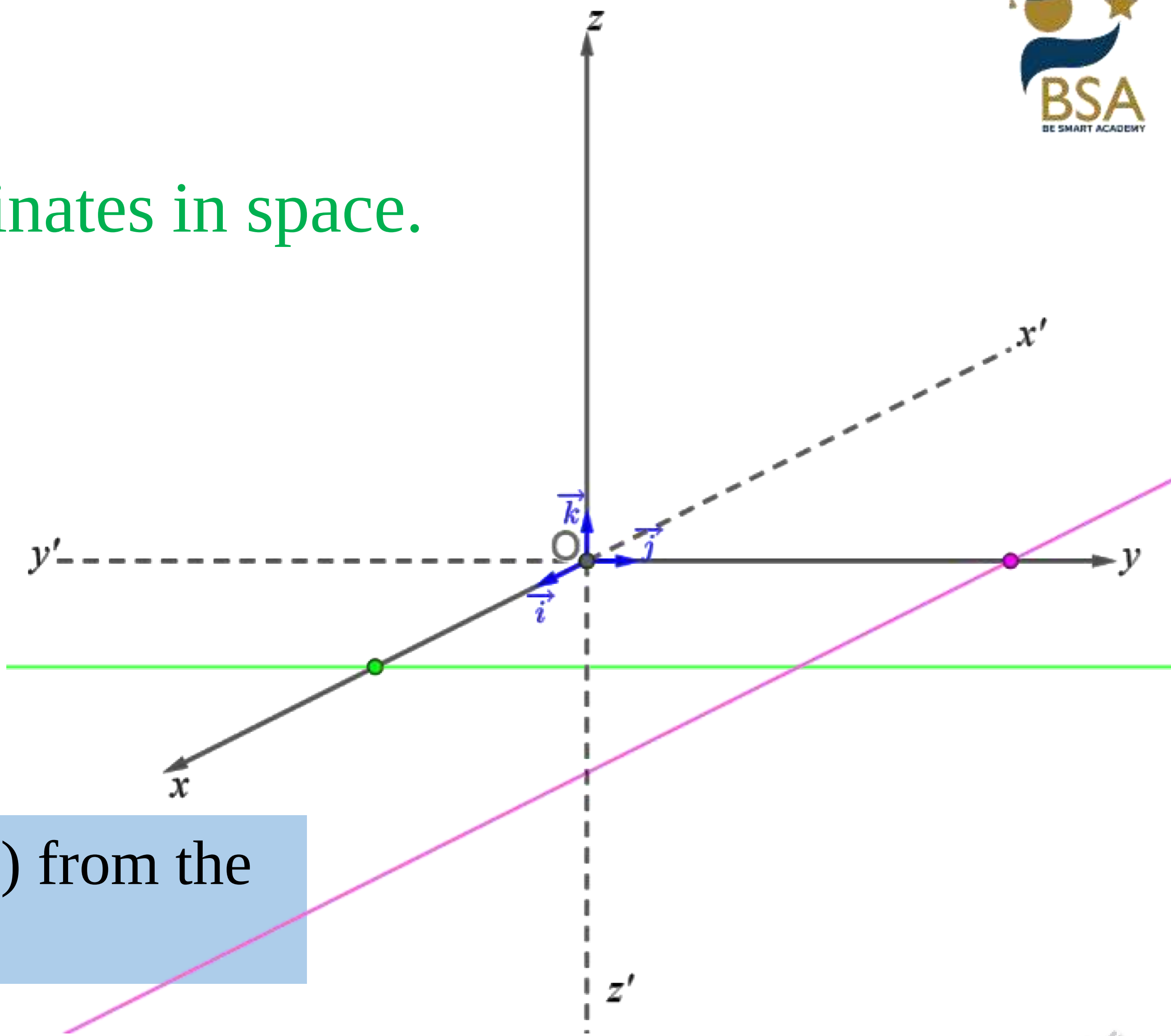
Step 3: Plot on ($y'y$) the point of ordinate 4.

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



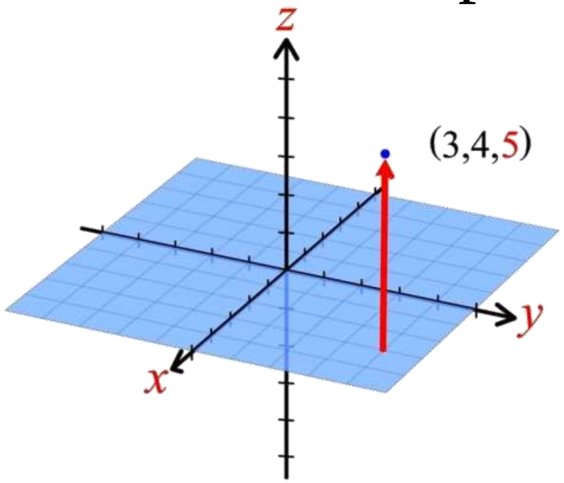


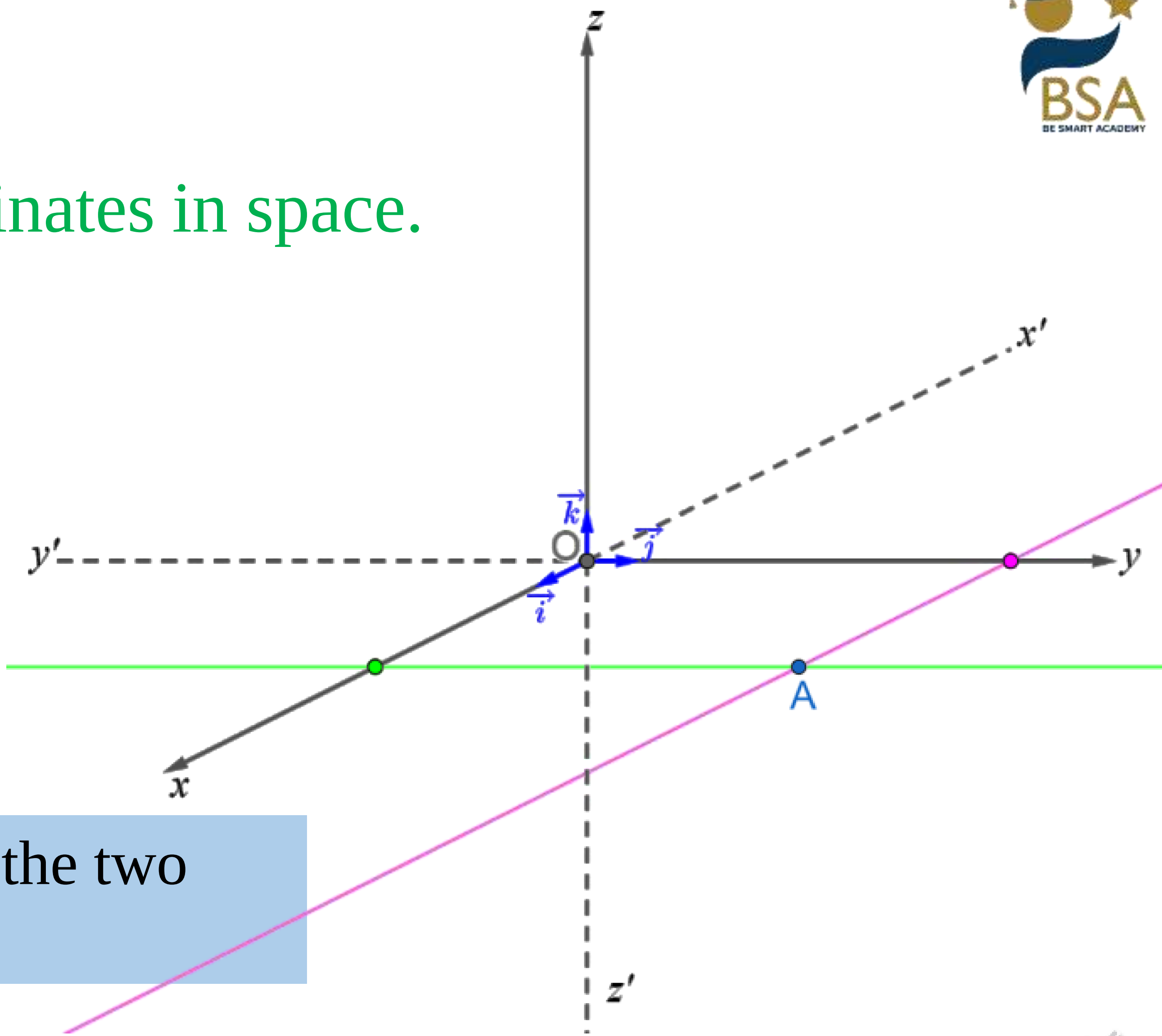
Step 4: Draw the parallel line to ($x'x$) from the previous point.

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



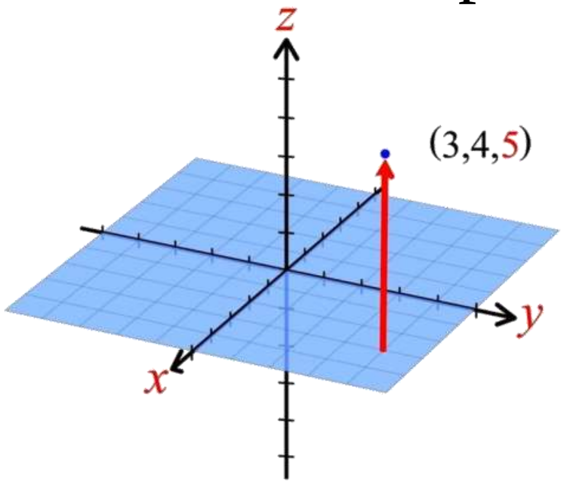


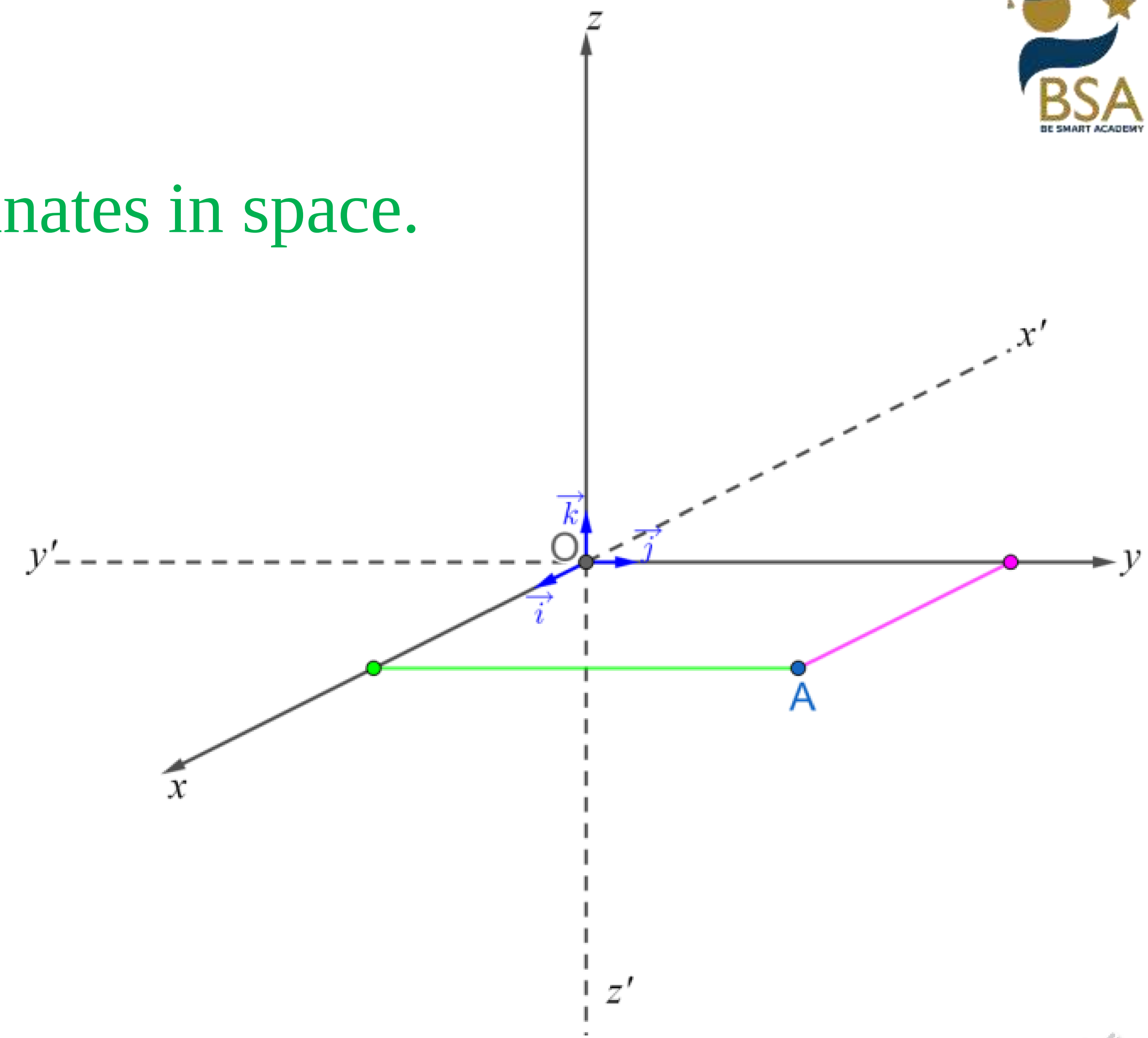
Step 5: Plot the intersection point of the two drawn parallel.

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



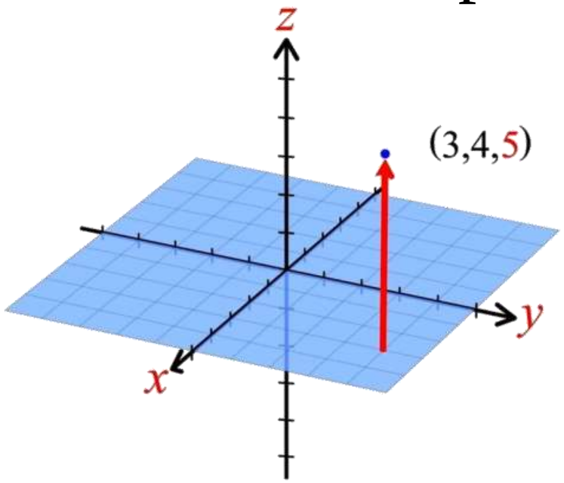


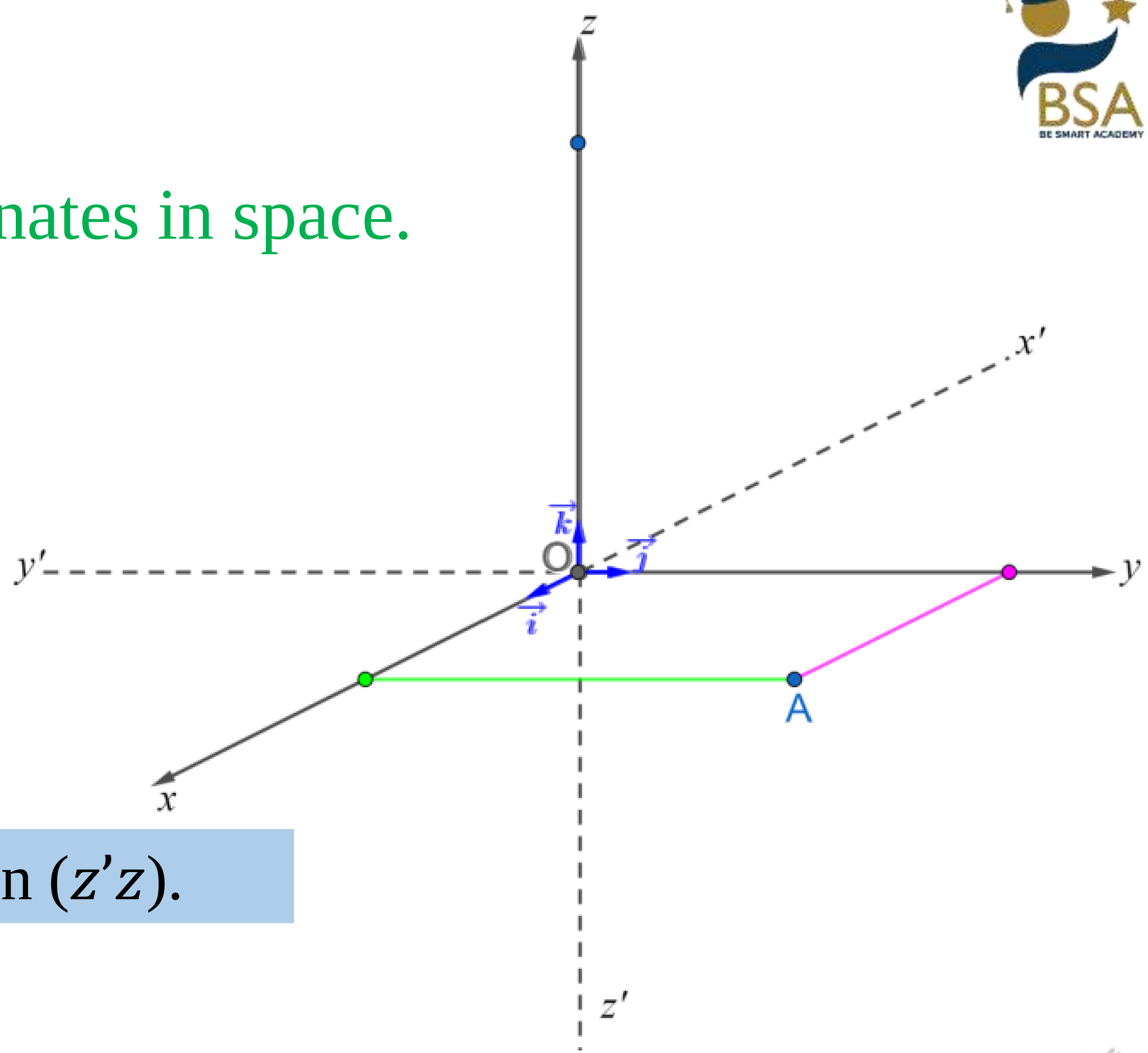
We get the point A(3;4;0)

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



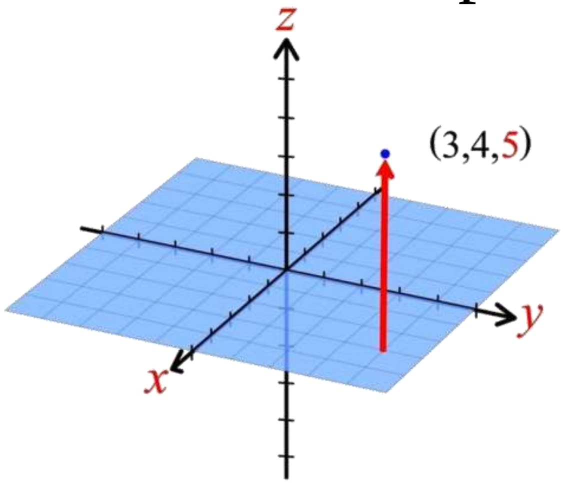


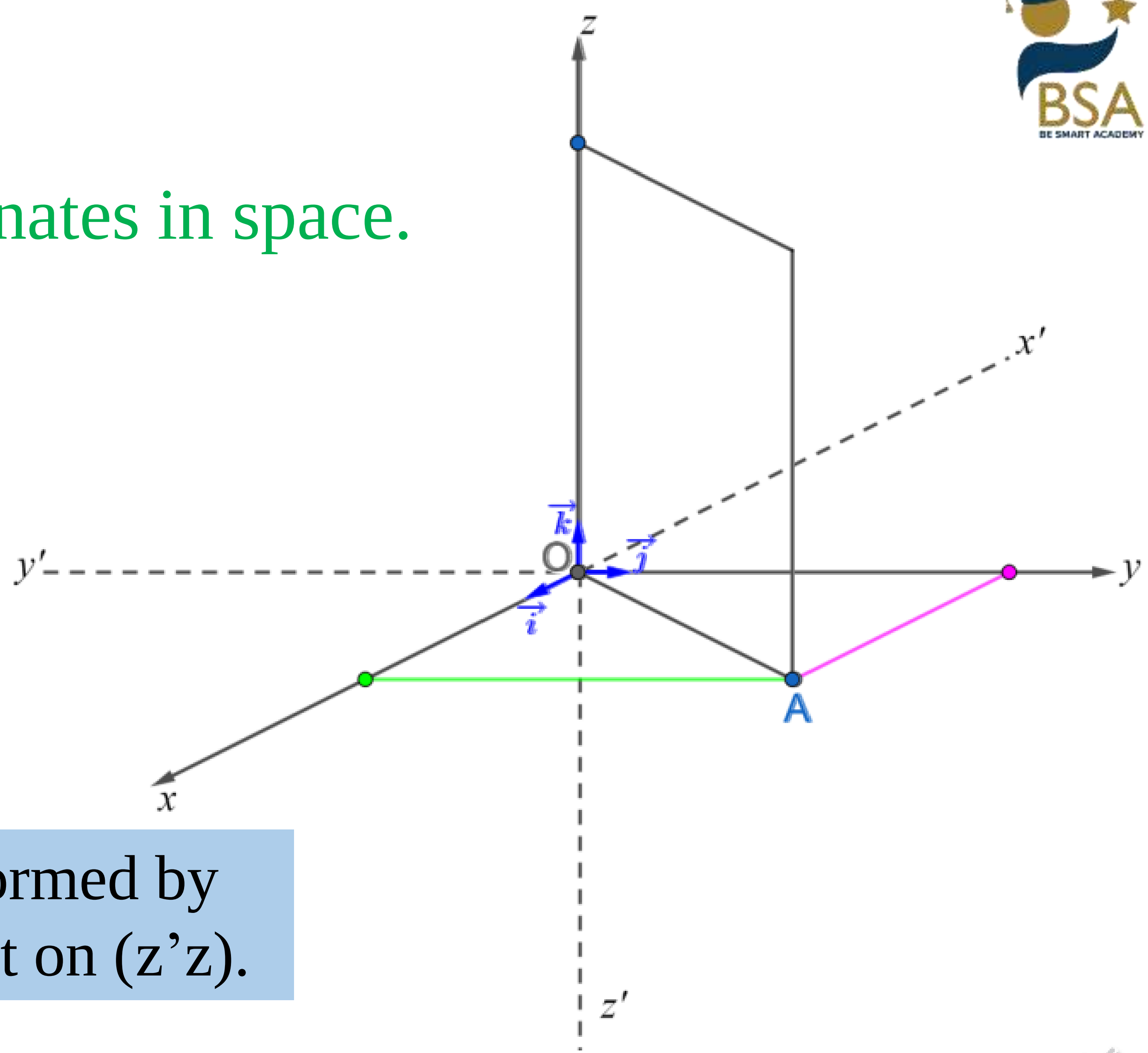
Step 6: Plot the point of elevation 5 on ($z'z$).

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)



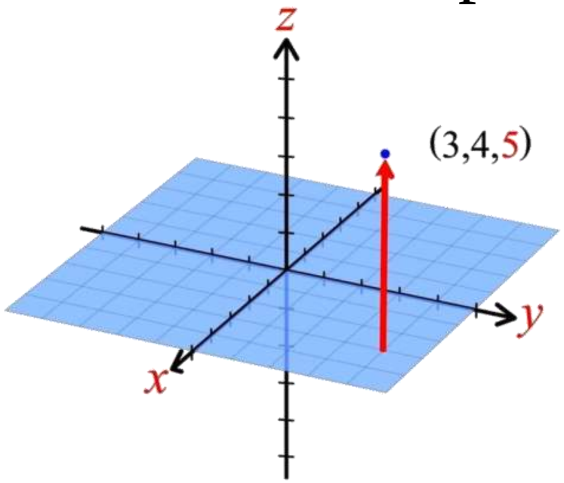


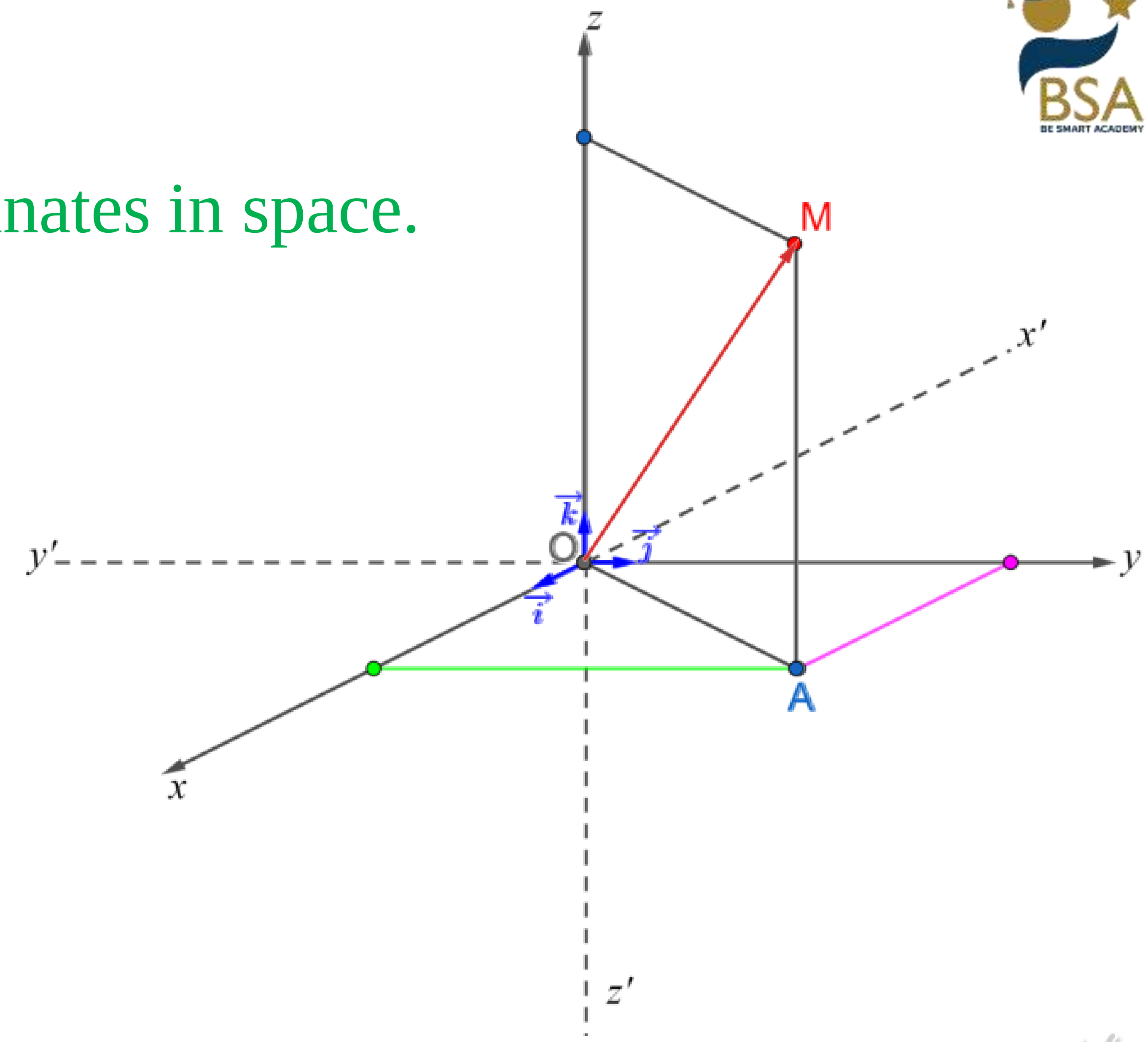
Step 6: complete the parallelogram formed by the points A, O and the previous point on (z’z).

System of coordinates
Coordinates of a point

Locating a point of given coordinates in space.

Consider the point M(3;4;5)





We get the point M(3;4;5)

System of coordinates
Analytic expressions

Coordinates of a vector.

Consider a vector $\vec{u}$ in an orthonormal system $(O; \vec{\imath}; \vec{\jmath}; \vec{k})$.

$$\vec{u} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

$$\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2}$$

Consider the two points A and B in an orthonormal system $(O; \vec{\imath}; \vec{\jmath}; \vec{k})$.

$$\overrightarrow{AB} \left| \begin{array}{l} x_{\overrightarrow{AB}} = x_B - x_A \\ y_{\overrightarrow{AB}} = y_B - y_A \\ y_{\overrightarrow{AB}} = z_B - z_A \end{array} \right.$$

$$\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

System of coordinates
Analytic expressions

Vector relations.

Consider the vector $\vec{u}$, $\vec{v}$ and $\vec{w}$ in an orthonormal system $(O;\vec{i}\,;\vec{j}\,;\vec{k})$.
If $\vec{w} = a\vec{u} + b\vec{v}$ where $a , b \in \mathbb{R}$, then:

$$\begin{cases} x_{\vec{w}} = ax_{\vec{u}} + bx_{\vec{v}} \\ y_{\vec{w}} = ay_{\vec{u}} + by_{\vec{v}} \\ z_{\vec{w}} = az_{\vec{u}} + bz_{\vec{v}} \end{cases}$$

As a result:

- ✓ If $\vec{u} = \vec{v}$, then $x_{\vec{u}} = x_{\vec{v}}$; $y_{\vec{u}} = y_{\vec{v}}$; $z_{\vec{u}} = z_{\vec{v}}$
- ✓ If $\vec{u} = a\vec{v}$, then $x_{\vec{u}} = ax_{\vec{v}}$; $y_{\vec{u}} = ay_{\vec{v}}$; $z_{\vec{u}} = az_{\vec{v}}$

System of coordinates
Analytic expressions

Example

Find the coordinates of $\vec{w}$ in each case:

❶ $\vec{w} = 2\vec{u}$; $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$

$$\vec{w} = 2\vec{u} = 2\left(2\vec{i} + 3\vec{j} - \vec{k}\right) = 4\vec{i} + 6\vec{j} - 2\vec{k}$$

❷ $\vec{w} = -\vec{u} + 3\vec{v}$; $\vec{u} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{v} = -\vec{j} + 2\vec{k}$

$$\vec{w} = -\vec{u} + 3\vec{v} = -\left(2\vec{i} + 3\vec{j} - \vec{k}\right) + 3\left(-\vec{j} + 2\vec{k}\right)$$
$$= -2\vec{i} - 3\vec{j} + \vec{k} - 3\vec{j} + 6\vec{k} = -2\vec{i} - 6\vec{j} + 7\vec{k}$$

System of coordinates
Analytic expressions

Collinear vectors.

Consider the vector $\vec{u}$ & $\vec{v}$ in an orthonormal system $(O; \vec{i}\,; \vec{j}\,; \vec{k})$.
$\vec{u}$ and $\vec{v}$ are collinear vector if: exist a real number k ≠ 0 such that $\vec{u} = k\vec{v}$

Example 1

$\vec{u}(2; -3; 4)$ and $\vec{v}(10; -15; 20)$

$x_{\vec{u}} = kx_{\vec{v}}$; $2 = 10k$; $k = \frac{2}{10} = \frac{1}{5}$

$y_{\vec{u}} = ky_{\vec{v}}$; $-3 = -15k$; $k = \frac{-3}{-15} = \frac{1}{5}$

$z_{\vec{u}} = kz_{\vec{v}}$; $4 = 20k$; $k = \frac{4}{20} = \frac{1}{5}$

So k = $\frac{1}{5}$ $\vec{u} = \frac{1}{5}\vec{v}$

Then $\vec{u}$ & $\vec{v}$ are collinear vectors.

System of coordinates
Analytic expressions

Collinear vectors.

Consider the vector $\vec{u}$ & $\vec{v}$ in an orthonormal system $(O; \vec{i}\,; \vec{j}\,; \vec{k})$.
$\vec{u}$ and $\vec{v}$ are collinear vector if: exist a real number k $\neq 0$ such that $\vec{u} = k\vec{v}$

Example 2

$\vec{u}(1; 0; -4)$ and $\vec{v}(2; -1; 5)$

$x_{\vec{u}} = kx_{\vec{v}}$; $1 = 2k$; $k = \frac{1}{2}$

$y_{\vec{u}} = ky_{\vec{v}}$; $0 = -1k$; $k = 0$

So k doesn't exist

Then $\vec{u}$ & $\vec{v}$ are not collinear vectors.

System of coordinates
Analytic expressions

Collinear points.

To show that three points A, B and C are collinear, it is sufficient to show that two vectors from these points are collinear.

Example

A(1;-1;2), B(2;0;4) and C(0;-2;0)

$\overrightarrow{AB}(1;1;2)$; $\overrightarrow{AC}(-1;-1;-2)$

$x_{\overrightarrow{AB}} = kx_{\overrightarrow{AC}}$; $1 = -k$; $k = -1$

$y_{\overrightarrow{AB}} = ky_{\overrightarrow{AC}}$; $1 = -1k$; $k = -1$

$z_{\overrightarrow{AB}} = kz_{\overrightarrow{AC}}$; $2 = -2k$; $k = -1$

So k = -1

Then A, B and C are collinear points.

System of coordinates
Analytic expressions

Coplanar vectors.
Example

$\vec{u}(1;-1;2)$, $\vec{v}(2;0;4)$ and $\vec{w}(0;-2;0)$

Exist a & b such that $\vec{w} = a\vec{u} + b\vec{v}$

$x_{\vec{w}} = ax_{\vec{u}} + bx_{\vec{v}}$

$0 = a + 2b$ (1)

$y_{\vec{w}} = ay_{\vec{u}} + by_{\vec{v}}$

$-2 = -a$ (2)

$z_{\vec{w}} = az_{\vec{u}} + bz_{\vec{v}}$

$0 = 2a + 4b$ (3)

(2): $a = 1$

(1): $0 = 1 + 2b$ so $b = -\frac{1}{2}$

Verification:

(3): $0 = 2(1) + 4\left(-\frac{1}{2}\right)$

$0 = 2 - 2$

$0 = 0$ true

So $a = 1; b = -\frac{1}{2}$

$\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar

System of coordinates
Analytic expressions

Coplanar points.

To show that four points A, B, C and D are coplanar, it is sufficient to show that three vectors from these points are coplanar.

Remark

Four points form a tetrahedron means that these points are not coplanar.

System of coordinates
Analytic expressions

Midpoint of a segment.

I is the midpoint of [AB]:

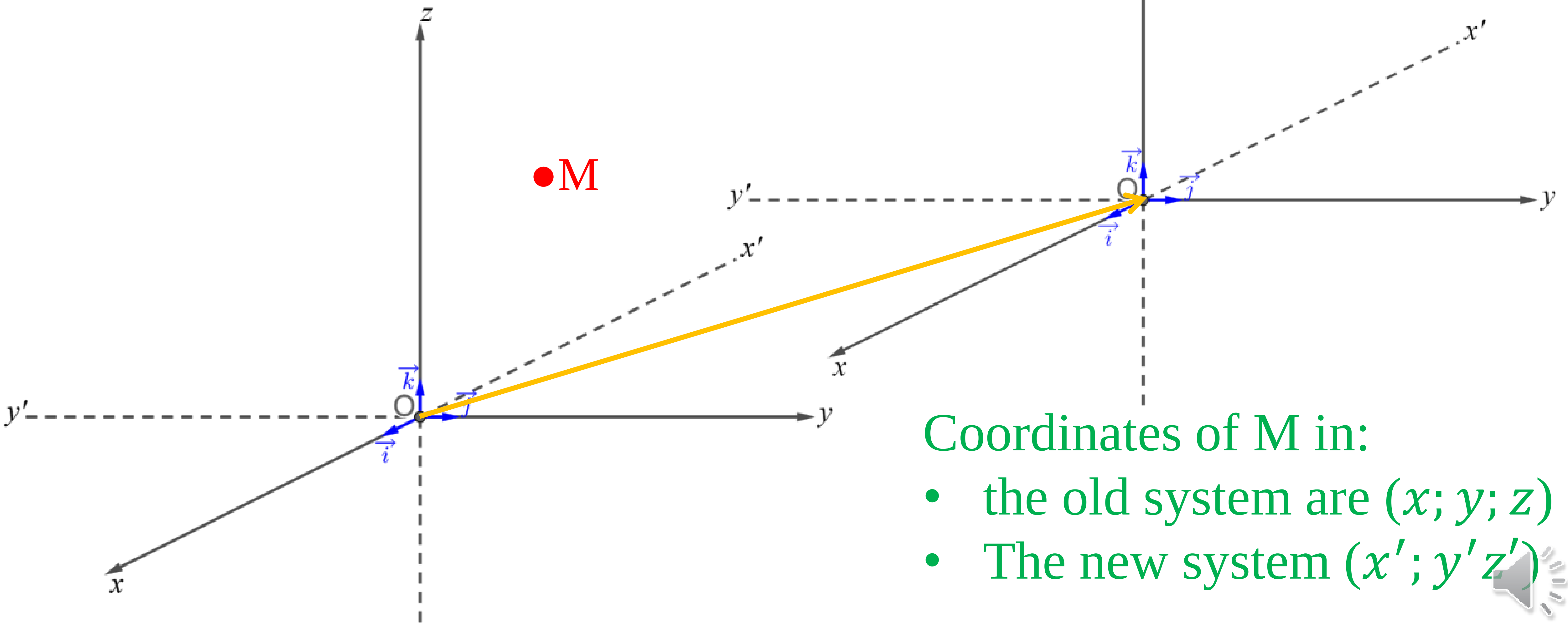
$$\begin{cases} x_I = \frac{x_A+x_B}{2} \\ y_I = \frac{y_A+y_B}{2} \\ z_I = \frac{z_A+z_B}{2} \end{cases}$$

Center of gravity of a triangle

G is the centroid of the triangle ABC:

$$\begin{cases} x_G = \frac{x_A+x_B+x_C}{3} \\ y_I = \frac{y_A+y_B+y_C}{3} \\ z_I = \frac{z_A+z_B+z_C}{3} \end{cases}$$

System of coordinates
Transformation of systems by translation



Coordinates of M in:

- the old system are $(x; y; z)$
- The new system $(x'; y'z')$

System of coordinates
Transformation of systems by translation

$$\left(O;\vec{i};\vec{j};\vec{k}\right) \xrightarrow{\overrightarrow{OO'}(a;b;c)} \left(O';\vec{i};\vec{j};\vec{k}\right)$$

$$(x;y;z) \qquad\qquad (x';y';z')$$

$$x = a + x' \quad ; \quad x' = x - a$$
$$y = b + y' \quad ; \quad y' = y - b$$
$$z = c + z' \quad ; \quad z' = z - c$$

Find the new coordinates of M(1;-1;3) in the new system $\left(O';\vec{i};\vec{j};\vec{k}\right)$ where O’(2;0;4).

$x' = x - 2 = 1 - 2 = -1$ $\qquad$ $y' = y - 0 = -1$

$z' = z - 4 = 3 - 4 = -1$